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A TRUNCATION ERROR REDUCING SCHEME FOR BALANCED FORECAST MODELS

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ABSTRACT

Various ways to improve the numerical accuracy of solutions to balanced forecast equations are discussed and compared. Among these, the most efficient method seems to be a correction operator technique with an associated space-truncation error of fourth order in Δs. Results from a number of real data short-range forecasts with this method are also presented and discussed.

1. INTRODUCTION

In balanced models for routine numerical weather prediction, commonly used values for the horizontal grid distance and the time step are $\Delta s \approx 300$–400 km, $\Delta t \approx 0.5$–1 hr. With this choice, the numerical errors are dominated by the space-truncation error, defined as the Δt-independent part of the total truncation error. (The remaining part is called the time-truncation error.) Its influence on the solution is often noticeable in 24-hr forecasts and may be considerable for extended forecast ranges.

As a simple example, the Rossby wave solution $\psi = -Uy + \psi' e^{ik(x-ct)}$ to the finite-difference form of the divergent barotropic vorticity equation

$$\frac{\partial}{\partial t}(\nabla^2\psi - q\psi) = -J(\psi, \nabla^2\psi + f) \tag{1}$$

with the 5-point approximations

$$\nabla^2_{(5)}\psi_{ij} = \frac{1}{\Delta s^2}(\psi_{i+1\,j} + \psi_{i-1\,j} + \psi_{i\,j+1} + \psi_{i\,j-1} - 4\psi_{ij}) \tag{2}$$

$$J_{(5)}(\alpha, \beta)_{ij} = J^{++}(\alpha, \beta)_{ij} = \frac{1}{4\Delta s^2}\Big((\alpha_{i+1\,j} - \alpha_{i-1\,j})(\beta_{i\,j+1} - \beta_{i\,j-1}) - (\alpha_{i\,j+1} - \alpha_{i\,j-1})(\beta_{i+1\,j} - \beta_{i-1\,j})\Big) \tag{3}$$

to the Laplace and Jacobi operators, moves with a phase speed

$$c = \delta \frac{k^2\gamma U - \beta}{k^2\gamma + q}$$

where

$$\gamma = \frac{2(1-\cos k_1\Delta s) + 2(1-\cos k_2\Delta s)}{(k\Delta s)^2},$$

$$\delta = \frac{k_1\Delta s\ \sin k_1\Delta s + k_2\Delta s\ \sin k_2\Delta s}{(k\Delta s)^2},$$

$k_1 = k\cos\varphi$, $k_2 = k\sin\varphi$, and φ is the angle between the x-axis and the i-direction. This speed may be much less than its correct value, given by

$$c = \frac{k^2U - \beta}{k^2 + q}.$$

For a grid oriented along the x- and y-axes, assuming $\Delta s = 300$ km, $U = 18$ m sec^{-1}, $q = 0.8 \cdot 10^{-12}$ m^{-2}, $\beta = 1.62 \cdot 10^{-11}$ m^{-1} sec^{-1}, the error is 10 percent for a wavelength $\lambda = 2\pi/k$ of 2600 km, 15 percent for $\lambda = 2000$ km, and 20 percent for $\lambda = 1800$ km (see fig. 1–3). The time truncation error is here neglected.

For other orientations of the grid the phase-speed errors are smaller, reaching a minimum (about half of the maximum value) when the grid is at a 45° angle to the x-y axes.

This directional dependence of the error causes a spurious distortion of the solution pattern, in addition to the reduction of the phase speeds. To avoid this type of error, slightly more complicated finite-difference approximations to the Laplace and Jacobi operators have to be

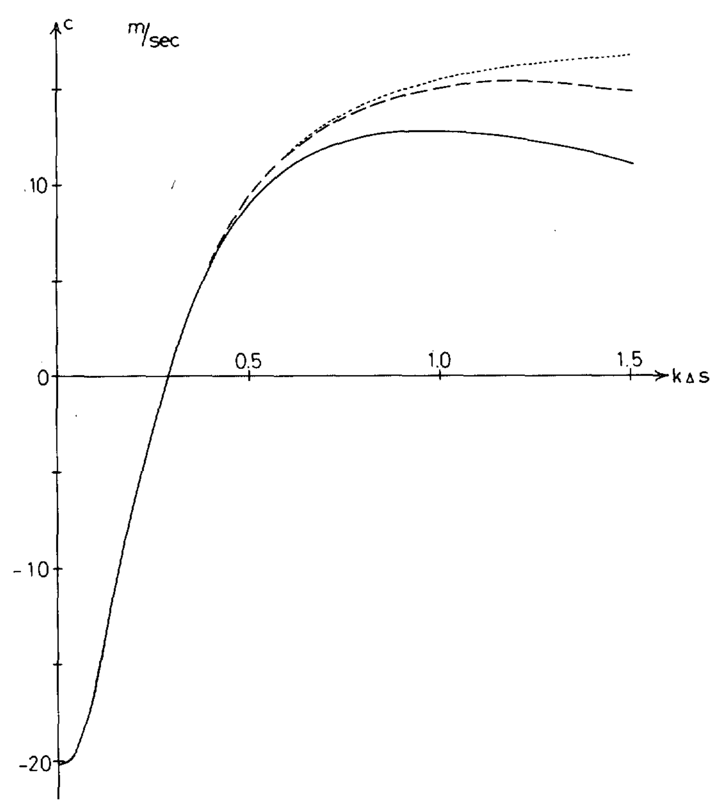

FIGURE 1.—Phase speed for a Rossby-wave solution to the divergent barotropic vorticity equation. Dotted line is for the exact equation, solid line for the second-order approximation, and dashed line for the new fourth-order method. $\Delta s = 300$ km, $U = 18$m sec^{-1}, $q = 0.8 \cdot 10^{-12}$ m^{-2}, $\beta = 1.62 \cdot 10^{-11}$ m^{-1} sec^{-1}.

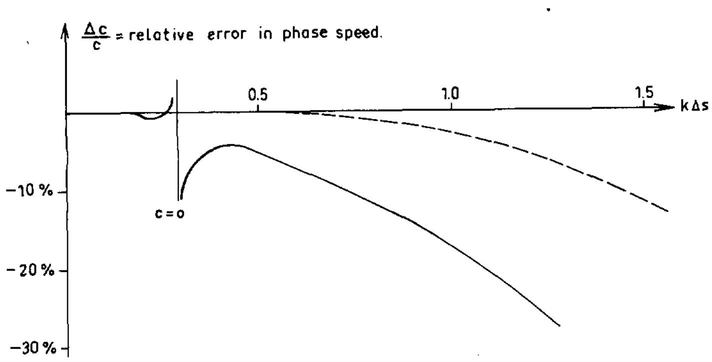

FIGURE 2.—Relative phase-speed error for the second-order method (solid line) and for the new fourth-order method (dashed line).

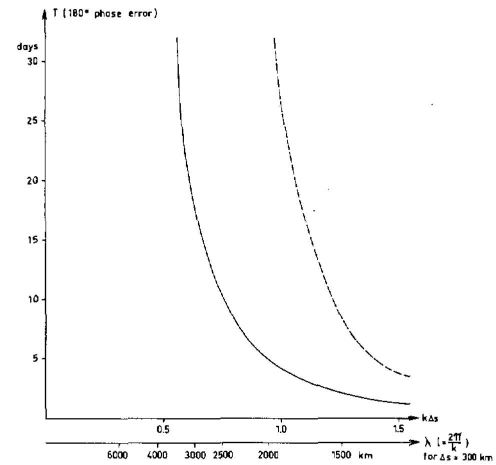

FIGURE 3.—Time for obtaining a phase error of 180°. Solid line shows values for the usual second-order method, dashed line values for the new method.

used. Of all possible expressions of the values at the point (i, j) and the eight adjoining points, the Laplacian with minimum orientational truncation error is (Miyakoda, 1960)

$$\nabla^2_{(9)}\psi_{ij}=\frac{1}{6\Delta s^2}(\psi_{i+1\,j+1}+\psi_{i+1\,j-1}+\psi_{i-1\,j+1}+\psi_{i-1\,j-1}+4(\psi_{i+1\,j}+\psi_{i-1\,j}+\psi_{i\,j+1}+\psi_{i\,j-1})-20\psi_{ij}). \tag{4}$$

The corresponding Jacobian is not uniquely defined, and any member of the one-parameter set

$$\begin{aligned}J^{a}_{(9)}(\alpha,\beta)_{ij}=\frac{1}{4\Delta s^2}\Big\{&\frac{a}{3}[(\alpha_{i+1\,j}-\alpha_{i-1\,j})(\beta_{i\,j+1}-\beta_{i\,j-1})-(\alpha_{i\,j+1}-\alpha_{i\,j-1})(\beta_{i+1\,j}-\beta_{i-1\,j})]\\
&+\frac{2-a}{3}[\alpha_{i+1\,j}(\beta_{i+1\,j+1}-\beta_{i+1\,j-1})-\alpha_{i-1\,j}(\beta_{i-1\,j+1}-\beta_{i-1\,j-1})-\alpha_{i\,j+1}(\beta_{i+1\,j+1}-\beta_{i-1\,j+1})+\alpha_{i\,j-1}(\beta_{i+1\,j-1}-\beta_{i-1\,j-1})]\\
&+\frac{2-a}{3}[\beta_{i\,j+1}(\alpha_{i+1\,j+1}-\alpha_{i-1\,j+1})-\beta_{i\,j-1}(\alpha_{i+1\,j-1}-\alpha_{i-1\,j-1})-\beta_{i+1\,j}(\alpha_{i+1\,j+1}-\alpha_{i+1\,j-1})+\beta_{i+j}(\alpha_{i-1\,j+1}-\alpha_{i-1\,j-1})]\\
&+\frac{a-1}{3}\cdot\frac{1}{2}[(\alpha_{i+1\,j+1}-\alpha_{i-1\,j-1})(\beta_{i-1\,j+1}-\beta_{i+1\,j-1})-(\alpha_{i-1\,j+1}-\alpha_{i+1\,j-1})(\beta_{i+1\,j+1}-\beta_{i-1\,j-1})]\Big\}\\
&=\frac{a}{3}J^{++}(\alpha,\beta)_{ij}+\frac{2-a}{3}J^{+\times}(\alpha,\beta)_{ij}+\frac{2-a}{3}J^{\times+}(\alpha,\beta)_{ij}+\frac{a-1}{3}J^{\times\times}(\alpha,\beta)_{ij}\end{aligned} \tag{5}$$

may be used. Although there is no restriction on the value of a, the cases $a=0$, 1, or 2 are of main interest. For $a=1$ we have the symmetric conservative Jacobian proposed by Arakawa (1966) to guarantee computational stability. (This property has been confirmed by Sundström, 1969.)

The approximate independence of the grid orientation follows from the Taylor series expansions

$$\nabla^2_{(9)}\psi=\nabla^2\psi+\frac{\Delta s^2}{12}\nabla^4\psi+0(\Delta s^4) \tag{6}$$

$$J^{a}_{(9)}(\alpha,\beta)=J(\alpha,\beta)+\frac{\Delta s^2}{12}\Big\{aJ(\alpha,\nabla^2\beta)+aJ(\nabla^2\alpha,\beta)+(2-a)\nabla^2J(\alpha,\beta)\Big\}+0(\Delta s^4). \tag{7}$$

Unfortunately, the maximum phase-speed error still remains the same. On the average, the phase-speed error may thus actually increase, and it is mainly in combination with one of the following error-reducing methods that the orientation-independent operators are of importance.

2. POSSIBLE WAYS TO IMPROVE THE NUMERICAL ACCURACY OF THE FINITE-DIFFERENCE APPROXIMATIONS

1) The simplest way to diminish the truncation errors is of course simply refining the grid, which should obviously be done in such a way that the total computation time is kept as small as possible. Since we cannot *a priori* tell whether Δs and Δt should both be reduced, we must first seek for the choice giving the smallest possible total truncation error (E) within a given total computation time (τ), or alternatively, the smallest τ for a given error E. If the leap-frog method is used for the time differencing, E is of second order in Δs and Δt while τ is approximately proportional to the number of time steps ($T/\Delta t$), the number of grid points in the horizontal ($S/\Delta s^2$), and the number of iterations used for inverting the Helmholz operator at each time step. For the extrapolated Liebmann method, the optimal choice of Δs and Δt is then the one making the space truncation error three times as large as the time truncation error (if the choice is not in conflict with the stability conditions). The proof is found in the Appendix (Case 1). It is also shown there that the minimum compu-

tation time is proportional to E^{-2}—to reduce the error by a factor of two, the computation time increases by a factor of four. For the Alternating-Direction Implicit (ADI) method the relation is slightly more favorable and τ is nearly proportional to $E^{-3/2}$. Anyhow, refining the grid is only possible if the memory capacity of the computer is abundant, since the utilized capacity increases as E^{-1}.

2) If the computation is made with two different grid distances, one may extrapolate the results to zero grid distance (the Richardson-extrapolation technique, see Bulirsch and Stoer, 1966). To facilitate the programming, the larger grid distance ($\Delta s''$) is usually chosen as a multiple of the smaller one ($\Delta s'$). The most favorable case is $\Delta s''=2\Delta s'$, $\Delta t''=2\Delta t'$. The minimum total computation time is then proportional to E^{-1}, if the extrapolated Liebmann method is used, and nearly to $E^{-3/4}$ for the ADI method (see the Appendix, Case 2). In spite of these favorable results, the method is presently of limited interest. The asymptotic error formula is not valid for $k\Delta s$ larger than about 1.5, giving an upper bound for Δs around 400 km. With $\Delta s'=200$ km, $\Delta s''=400$ km, the computation time is five times as large as for the conventional scheme with $\Delta s=300$ km, and the gain in accuracy hardly justifies such an increase.

3) The two remaining methods are based on approximations with a local space truncation error of fourth order in Δs (but where the usual leap-frog scheme is used for the time differencing).

a) The easiest way to obtain such a scheme is inserting fourth-order approximations to the Laplace and Jacobi operators in the forecast equation. These finite-difference operators must use values at points outside the central 9-point set and may be constructed from the previously given 5- or 9-point formulae as

$$\nabla_*^2\psi_{ij}=\frac{4}{3}\nabla^2(\Delta s)\psi_{ij}-\frac{1}{3}\nabla^2(2\Delta s)\psi_{ij} \tag{8}$$

and

$$J_*(\alpha,\beta)_{ij}=\frac{4}{3}J(\Delta s)(\alpha,\beta)_{ij}-\frac{1}{3}J(2\Delta s)(\alpha,\beta)_{ij}. \tag{9}$$

Schemes of this type have been thoroughly tested by Miyakoda (1960, 1962), showing that they accomplish the desired error reduction as long as the shortest wavelength exceeds $4\Delta s$. As shown in the Appendix (Case 3), the total computation time is proportional to $E^{-5/4}$ with the optimum choice of Δs and Δt, provided that the number of iterations used for inverting the Helmholz operator with the extrapolated Liebmann method is proportional to Δs^{-1}. This is, however, not strictly true, and the computation time may therefore increase more rapidly. Futhermore, solution values must be prescribed both at the boundary and at grid points just outside it, which may cause stability problems. To avoid these difficulties, Miyakoda investigated a scheme with a fourth-order Jacobian but with a conventional Laplace operator and found that it gave a noticeable improvement of the phase-speed errors; such a simplified scheme was introduced in the barotropic forecasting routine of the Japan Meteorological Agency in 1960. This change increased the total computation time by only about 20 percent for short-range forecasts and even less for longer integration times.

b) By a correction-operator technique, showing some similarity to Thompson's (1955) inverse-averaging procedure, it is however possible to find a fourth-order scheme where the Helmholz operator is only a 9-point expression, which may be inverted with the usual extrapolated Liebmann method without any boundary value problems. The derivation of this method is described in section 3, and the associated phase-speed error for a Rossby-wave solution is discussed in section 4 of this paper. It has been successfully tested for a number of cases both with a simple channel-flow model and the 24-hr and 48-hr barotropic forecast routine of the Swedish Meterological and Hydrological Institute. Some of these integrations were extended to 72-hr and 96-hr forecast time. The average reduction of the root-mean-square error in 24-hr forecasts was about 10 percent. In the 48-hr forecast, the improvement was less than this value and it even failed to appear in the 72-hr forecasts. The reason for this was found to be the inability of the barotropic scheme to predict the motion of planetary-scale systems, causing an erroneous retrogression of the waves. This retrogression is partly reduced by the influence of the semiempiric divergence parameter q, but also by the phase-speed errors, which act in the same direction. With the new scheme, a larger optimum value of q should therefore be expected. This value depends upon the size of the forecast region, the length of the forecast, and the desired accuracy for different scales of motion. After such a modification, the correction-operator technique is probably the most efficient method of reducing the numerical errors of balanced forecast schemes.

3. DERIVATION OF THE CORRECTION-OPERATOR SCHEME

From the Taylor series expansions (6, 7), it follows that

$$\nabla^2\psi=\left(1-\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)\nabla^2_{(9)}\psi+0(\Delta s^4) \tag{10}$$

and

$$J(\alpha,\beta)=\left(1-(2-a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)J^{a}_{(9)}\left(\left(1-a\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)\alpha,\right.$$

$$\left.\left(1-a\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)\beta\right)+0(\Delta s^4) \tag{11}$$

(where a is the parameter in the Jacobian (5)) if the functions ψ, α, and β have derivatives of sufficiently high order. With the aid of the relations

$$\left(1+\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)(\nabla^2\psi-q\psi)=\left(1-q\frac{\Delta s^2}{12}\right)\nabla^2_{(9)}\psi-q\psi+0(\Delta s^4), \tag{12}$$

$$\left(1+\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)\left(1-(2-a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)J^a_{(9)}\left(\left(1-a\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)\psi,\left(1-a\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)(\nabla^2\psi+f)\right)$$
$$=\left(1-(1-a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)J^a_{(9)}\left(\left(1-a\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)\psi,\left(1-(1+a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)\left(\nabla^2_{(9)}\psi+\left(1+\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)f\right)\right)+0(\Delta s^4)$$
$$=\left(1-(1-a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)J^a_{(9)}\left(\psi+a\frac{\Delta s^2}{12}f,\left(1-(1+a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)\left(\nabla^2_{(9)}\left(\psi+a\frac{\Delta s^2}{12}f\right)+\left(1+(1-a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)f\right)\right)$$
$$+0(\Delta s^4)=\left(1-(1-a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)J^a_{(9)}\left(\tilde{\psi},\left(1-(1+a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)\left(\nabla^2_{(9)}\tilde{\psi}+\tilde{f}\right)\right)+0(\Delta s^4) \quad (13)$$

with

$$\tilde{\psi}=\psi+a\frac{\Delta s^2}{12}f \quad (14)$$

and

$$\tilde{f}=\left(1+(1-a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)f, \quad (15)$$

it may be shown that the equation

$$\frac{\partial}{\partial t}\left[\left(1-q\frac{\Delta s^2}{12}\right)\nabla^2_{(9)}\tilde{\psi}-q\tilde{\psi}\right]=-\left(1-(1-a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)\times J^a_{(9)}\left(\tilde{\psi},\left(1-(1+a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)\left(\nabla^2_{(9)}\tilde{\psi}+\tilde{f}\right)\right) \quad (16)$$

gives an approximation to equation (1) with a total truncation error $0(\Delta t^2)+0(\Delta s^4)$, if the leap-frog method is used for the time differencing. A comparison with the normal scheme based on 9-point operators shows that the only new features are the two correction operators $1-(1-a)\frac{\Delta s^2}{12}\nabla^2_{(9)}$ and $1-(1+a)\frac{\Delta s^2}{12}\nabla^2_{(9)}$, the factor $1-q\frac{\Delta s^2}{12}$ in front of the Laplace operator, and the use of $\tilde{\psi},\tilde{f}$ instead of ψ, f. For that scheme, one usually assumes ψ and $\nabla^2_{(9)}\psi$ to be prescribed at the boundary points. Here, additional conditions must be imposed, one if $a=-1$ or 1 and two for other choices of a.

1) If $a\neq-1$, one may assume that $\nabla^4_{(9)}\psi$ is also given at the boundary points, when computing the corrected vorticity $\left(1-(1+a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)\left(\nabla^2_{(9)}\tilde{\psi}+\tilde{f}\right)$.

2) If $a\neq 1$, the value of the right-hand side for points next to the boundary involves boundary-point values of

$$J^a_{(9)}\left(\tilde{\psi},\left(1-(1+a)\frac{\Delta s^2}{12}\nabla^2_{(9)}\right)\left(\nabla^2_{(9)}\tilde{\psi}+\tilde{f}\right)\right).$$

Since they occur in a term multiplied by Δs^2, only an error of order Δs^4 is committed if they are taken to be equal to $-\frac{\partial}{\partial t}\left[\nabla^2_{(9)}\tilde{\psi}-q\tilde{\psi}\right]$ (which is known at the boundary) as long as these values actually produce a smooth solution.

4. PHASE-SPEED ERRORS FOR THE NEW SCHEME

A simple measure of the accuracy of finite-difference approximations to the vorticity equation (1) is the phase-speed error for a Rossby-wave solution $\psi=-Uy+\psi' e^{ik(x-ct)}$. The magnitude of the errors for the usual second-order scheme with 5- or 9-point Laplace and Jacobi operators was discussed in the Introduction. By direct insertion in (4) and (5), the relations

$$\nabla^2_{(9)}e^{ik(x-ct)}=-k^2\gamma e^{ik(x-ct)} \quad (17)$$

and

$$J^a_{(9)}(y, e^{ik(x-ct)})=-ik\delta e^{ik(x-ct)} \quad (18)$$

where

$$\gamma=\frac{2(1-\cos k\,\Delta s)}{(k\,\Delta s)^2},\ \delta=\frac{\sin k\,\Delta s}{k\,\Delta s}$$

are easily found to be true for a grid oriented in the x- and y-directions, and the phase speed is consequently

$$c=\delta\left(1+\frac{1-a}{12}\gamma(k\,\Delta s)^2\right)\frac{k^2\gamma\left(U-a\frac{\Delta s^2}{12}\beta\right)\left(1+\frac{1+a}{12}\gamma(k\Delta s)^2\right)-\beta}{k^2\gamma+\left(1-\frac{\gamma}{12}(k\Delta s)^2\right)q}. \quad (19)$$

For small $k\,\Delta s$, this gives $c=\frac{k^2U-\beta}{k^2+q}+0((k\,\Delta s)^4)$, as expected.

The Rossby-wave phase speed was computed by this formula for the case $a=0$ with $\Delta s=300$ km, $U=18$ m sec^{-1}, $q=0.8\cdot10^{-12}$m^{-2}, and $\beta=1.62\cdot10^{-11}$m^{-1} sec^{-1} (corresponding to 45° lat.); see figure 1. The relative phase-speed error and the time required for obtaining a phase error of 180° are also shown in figures 2 and 3. For $k\Delta s<0.75$, the error is always less than 1 percent, and for larger values of $k\Delta s$, the error is always negative and approximately 25 percent of the error of the conventional scheme. In this wavelength range, U is much larger than β/k^2, and it may then be shown that $a=0$ gives the smallest phase-speed error. Although the Jacobian with $a=1$ has the merit of giving computational stability with the usual second-order scheme, no corresponding stability theorem has been found for the fourth-order method. The value $a=0$ was therefore used in the experiments, and the results showed no sign of instability provided the initial field was sufficiently smooth and Δt was less than about $(1.5U)^{-1}\Delta s$.

5. APPENDIX

With a sufficiently fine grid, the total computation time τ for a numerical integration of the barotropic vorticity equation is approximately proportional to the number of time steps $(T/\Delta t)$, the number of iterations used for inverting the Laplace or Helmholz operator at each time step, and the number of grid points in the horizontal $(S/\Delta s^2)$. For the extrapolated Liebmann method, the asymptotic rate of convergence is of order $0(\Delta s)$. The number of iterations is consequently $0(\Delta s^{-1})$, so that

$$\tau_L\approx K_1\Delta s^{-3}\Delta t^{-1}.$$

For the ADI method, the rate of convergence is of order $(\log S/\Delta s^2)^{-1}$ and thus

$$\tau_{ADI}\approx K_2\Delta s^{-2}\log\,(S/\Delta s^2)\Delta t^{-1}.$$

1) For the conventional scheme, the total truncation error E is given by $E=A\,\Delta t^2+B\,\Delta s^2+$higher order terms; $A/B>0$. By the method of Lagrange multipliers, the minimum value of τ_L for a given E is found to be

$$(\tau_L)_{min}\approx\left(\frac{B}{3A}\right)^{3/2}16A^2K_1E^{-2} \tag{20}$$

with

$$E\approx\frac{4B}{3}\Delta s^2 \tag{21}$$

for sufficiently small E, provided that the value for $\Delta t/\Delta s=(B/3A)^{1/2}$ is not in conflict with the stability conditions. Otherwise, choose the largest possible $\Delta t/\Delta s=\chi$ giving

$$\tau_{L_{min}}\approx(A+B/\chi^2)^2\chi^3K_1E^{-2} \tag{22}$$

with

$$E\approx(A\chi^2+B)\Delta s^2. \tag{23}$$

Similarly,

$$\tau_{\mathrm{ADI}_{min}}\approx K_2\Delta s^{-3}\log S/\Delta s^2\left(\frac{2A}{B}\left(1+\frac{1}{\log S/\Delta s^2}\right)\right)^{1/2}$$

$$\approx\frac{B}{2A}(3A)^{3/2}K_2E^{-3/2}\log\left(\frac{3B}{2E}S\right) \tag{24}$$

with

$$E\approx B\Delta s^2\frac{1+\tfrac{3}{2}\log S/\Delta s^2}{1+\log S/\Delta s^2}$$

$$\approx\frac{3B}{2}\Delta s^2 \tag{25}$$

corresponding to (20, 21). Formulae analogous to (22, 23) are obtained in the same way.

2) To obtain the same formulae for the Richardson extrapolation method, the fourth-order terms in the expansion for E must be considered: $E=A_1\Delta t^2-A_2\Delta t^4+B_1\Delta s^2-B_2\Delta s^4-C_2\Delta t^2\Delta s^2+$ higher order terms, with A_2, B_2, and C_2 of the same sign. Assume now

$$\frac{\Delta t'}{\Delta s'}=\frac{\Delta t''}{\Delta s''}=\kappa.$$

Then
$$\tau_L\approx K_1\kappa^{-1}(\Delta s'^{-4}+\Delta s''^{-4})$$

and

$$E\approx\frac{\Delta s''^2E(\Delta s')-\Delta s'^2E(\Delta s'')}{\Delta s''^2-\Delta s'^2}=(A_2\kappa^4+B_2+C_2\kappa^2)\Delta s'^2\Delta s''^2,$$

neglecting the higher order terms. If $\Delta s''$ is a multiple of $\Delta s'$, $\Delta s''=n\,\Delta s'$,

$$\tau_L\approx K_1\kappa^{-1}(n^2+n^{-2})(A_2\kappa^4+B_2+C^2\kappa^2)E^{-1}$$

and

$$E\approx n^2(A_2\kappa^4+B_2+C_2\kappa^2)\Delta s'^4.$$

The optimal choice of κ is

$$\left(\left(\left(\frac{C_2}{6A_2}\right)^2+\frac{B_2}{3A_2}\right)^{1/2}-\frac{C_2}{6A_2}\right)^{1/2}=\kappa_{opt}$$

giving

$$\tau_{L_{min}}\approx\tfrac{2}{3}K_1\kappa_{opt}^{-1}(n^2+n^{-2})(2B_2+C_2\kappa_{opt}^2)E^{-1} \tag{26}$$

$$E\approx\tfrac{2}{3}n^2(2B_2+C_2\kappa_{opt}^2)\Delta s'^4. \tag{27}$$

If this is in conflict with the stability condition, choose the largest possible $\frac{\Delta t}{\Delta s}=\chi$. Then

$$\tau_{L_{min}}\approx K_1\chi^{-1}(n^2+n^{-2})(A_2\chi^4+B_2+C_2\chi^2)E^{-1} \tag{28}$$

$$E\approx n^2(A_2\chi^4+B_2+C_2\chi^2)\Delta s'^4. \tag{29}$$

For the ADI method,

$$\tau_{\mathrm{ADI}}\approx K_2\kappa^{-1}(\Delta s'^{-3}\log S/\Delta s'^2+\Delta s'^{-3}\log S/\Delta s''^2)$$

giving

$$(\tau_{\mathrm{ADI}})_{min}\approx K_2\kappa_{opt}^{-1}[(n^{3/2}+n^{-3/2})\log S/\Delta s'^2-n^{3/2}\log n^2](\tfrac{3}{4}(2B_2+C_2\kappa_{opt}^2))^{3/4}E^{-3/4} \tag{30}$$

$$E\approx\tfrac{3}{4}n^2(2B_2+C_2\kappa_{opt}^2)\Delta s'^4 \tag{31}$$

with κ_{opt} approximately given by

$$\left(\left(\left(\frac{C_2}{8A_2}\right)^2+\frac{B_2}{2A_2}\right)^{1/2}-\frac{C_2}{8A_2}\right)^{1/2}.$$

3) For a scheme with a fourth-order space truncation error but only a second-order time-truncation error

$$E=A\Delta t^2+B\Delta s^4+\text{higher order terms},\ A/B>0,$$

the optimal time-step is $\Delta t=\left(\frac{2B}{3A}\right)^{1/2}\Delta s^2$ if the extrapolated Liebmann method is used. For sufficiently small Δs, this is not in conflict with any stability condition of the type $\frac{\Delta t}{\Delta s}\leq\chi$. Then,

$$\tau_{L_{min}}\approx K_1\left(\frac{3A}{2B}\right)^{1/2}\left(\frac{5B}{3}\right)^{5/4}E^{-5/4} \tag{32}$$

and

$$E\approx\frac{5B}{3}\Delta s^4. \tag{33}$$

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